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A counterexample to the question proposed by Yanagi–Furuichi–Kuriyama on matrix inequalities and related counterexamples

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Dedicated to Professor Tadasi Huruya and Professor Sin-Ei Takahasi on their 60th birthdays with respect and affection

Abstract

We shall give an counterexample to some questions proposed by Yanagi–Furuichi–Kuriyama on matrix inequalities and we consider some problems on matrix inequalities related to the questions proposed by them and we shall give several counterexamples to these questions.

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1. Introduction

In this paper a capital letter means $n \times n$ matrix. Firstly we shall give an counterexample in Section 2 to some questions proposed by Yanagi–Furuichi–Kuriyama

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on trace inequalities. Secondly we consider some problems in Section 3 on matrix inequalities related to the questions proposed by Yanagi–Furuichi–Kuriyama and we shall give several counterexamples in Section 4 to thses problems in Section 3.

Very recently Yanagi, Furuichi and Kuriyama [2–Problem 2.1] proposed the following problem on the trace inequality to give a partial answer of the open problem conjectured by Holevo [1].

Problem. Prove

$$\mathrm{Tr}\left[(A+B)^s\left\{A(\log A)^2+B(\log B)^2\right\}-(A+B)^{s-1}(A\log A+B\log B)^2\right]\geqslant 0$$

for any $s\in[0,1]$ and any two positive matrices $A\leqslant I$ and $B\leqslant I$.

Moreover Yanagi, Furuichi and Kuriyama [2–Question 2.7] proposed the following question in the matrix inequality related to the problem stated above.

Question. We do not know whether the following matrix inequalities

$$(1)\quad (A+B)^{\frac{1}{2}}\left\{A(\log A)^2+B(\log B)^2\right\}(A+B)^{\frac{1}{2}}\geqslant(A\log A+B\log B)^2$$

or

$$(2)\quad \left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}(A+B)\left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}$$
$$\geqslant(A\log A+B\log B)^2$$

for any two positive matrices $A\leqslant I$ and $B\leqslant I$ hold or not. We have not yet found any counterexamples, namely the examples that matrix inequalities both (1) and (2) are not satisfied simultaneously, for some positive matrices $A\leqslant I$ and $B\leqslant I$.

Firstly we shall give a counterexample to this question in the next section.

2. Two positive matrices $A\leqslant I$ and $B\leqslant I$ which do not simultaneously satisfy (1) and (2)

We can construct two positive matrices $A\leqslant I$ and $B\leqslant I$ which do not simultaneously satisfy (1) and (2) as follows.

At first we take two self-adjoint matrices C and D as follows:

$$C=\begin{pmatrix}0.215542 & 0.253223 & 0.124734\\ 0.253223 & 0.167724 & 0.438143\\ 0.124734 & 0.438143 & 0.549844\end{pmatrix}$$

and

$$D = \begin{pmatrix} 0.428514 & 0.221754 & 0.173325 \\ 0.221754 & 0.453775 & 0.391242 \\ 0.173325 & 0.391242 & 0.312979 \end{pmatrix}.$$

Next we construct A and B as follows:

$$A = C^2 = \begin{pmatrix} 0.126139 & 0.151703 & 0.206418 \\ 0.151703 & 0.284223 & 0.345983 \\ 0.206418 & 0.345983 & 0.509856 \end{pmatrix} \tag{2.1}$$

and

$$B = D^2 = \begin{pmatrix} 0.262841 & 0.263463 & 0.215279 \\ 0.263463 & 0.408157 & 0.338422 \\ 0.215279 & 0.338422 & 0.281068 \end{pmatrix}. \tag{2.2}$$

Then A and B are both positive definite matrices since C and D are self-adjoint and $A = C^2$ and $B = D^2$, in fact the eigenvalues of A are $0.851426\ldots$, $0.0363208\ldots$ and $0.0324711\ldots$, and then the eigenvalues of B are $0.877323\ldots$, $0.074533\ldots$ and $0.000207318\ldots$, so A and B are both positive matrices $A \leqslant I$ and $B \leqslant I$.

Then A is diagonalized as follows:

$$A = U \begin{pmatrix} 0.851426\ldots & 0 & 0 \\ 0 & 0.0363208\ldots & 0 \\ 0 & 0 & 0.0324711\ldots \end{pmatrix} U^*,$$

where U is the following unitary matrix

$$U = \begin{pmatrix} -0.333029\ldots & -0.858727\ldots & 0.389461\ldots \\ -0.554339\ldots & -0.155815\ldots & -0.817576\ldots \\ -0.762758\ldots & 0.488170\ldots & 0.424135\ldots \end{pmatrix}.$$

Also B is diagonalized as follows:

$$B = V \begin{pmatrix} 0.877323\ldots & 0 & 0 \\ 0 & 0.0745353\ldots & 0 \\ 0 & 0 & 0.000207318\ldots \end{pmatrix} V^*,$$

where V is the following unitary matrix

$$V = \begin{pmatrix} -0.484388\ldots & -0.874437\ldots & 0.0269949\ldots \\ -0.674179\ldots & 0.353437\ldots & -0.648509\ldots \\ -0.557539\ldots & 0.33233\ldots & 0.760728\ldots \end{pmatrix}.$$

Then we have

$$\log A = \begin{pmatrix} -2.98250\ldots & 0.618034\ldots & 0.782807\ldots \\ 0.618034\ldots & -2.42090\ldots & 1.37267\ldots \\ 0.782807\ldots & 1.37267\ldots & -1.50022\ldots \end{pmatrix} \tag{2.3}$$

and

$$\log B = \begin{pmatrix} -2.02226\ldots & 0.90820\ldots & 0.545025\ldots \\ 0.90820\ldots & -3.95075\ldots & 3.82996\ldots \\ 0.545025\ldots & 3.82996\ldots & -5.23561\ldots \end{pmatrix} \tag{2.4}$$

and by using (2.1)–(2.4) we have

$$\begin{aligned} & A \log A + B \log B \\ & = \begin{pmatrix} -0.295789\ldots & 0.0163881\ldots & 0.0225023\ldots \\ 0.0163881\ldots & -0.196501\ldots & -0.0751776\ldots \\ 0.0225023\ldots & -0.0751776\ldots & -0.186477\ldots \end{pmatrix} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} & (A \log A + B \log B)^2 \\ & = \begin{pmatrix} 0.088266\ldots & -0.00975936\ldots & -0.0120841\ldots \\ -0.00975936\ldots & 0.0445329\ldots & 0.0291601\ldots \\ -0.0120841\ldots & 0.0291601\ldots & 0.0409316\ldots \end{pmatrix}. \end{aligned} \tag{2.6}$$

Moreover by using (2.1)–(2.4) we have

$$\begin{aligned} & A(\log A)^2 + B(\log B)^2 \\ & = \begin{pmatrix} 0.742459\ldots & -0.214626\ldots & -0.240415\ldots \\ -0.214626\ldots & 0.347301\ldots & -0.096009\ldots \\ -0.240415\ldots & -0.096009\ldots & 0.245371\ldots \end{pmatrix} \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} & \left\{A(\log A)^2 + B(\log B)^2\right\}^{\frac{1}{2}} \\ & = \begin{pmatrix} 0.815029\ldots & -0.179847\ldots & -0.214107\ldots \\ -0.179847\ldots & 0.543743\ldots & -0.138922\ldots \\ -0.214107\ldots & -0.1389822\ldots & 0.424534\ldots \end{pmatrix} \end{aligned} \tag{2.8}$$

and by using (2.1), (2.2), (2.6) and (2.7) we have

$$\begin{aligned} X &= (A+B)^{\frac{1}{2}} \left\{A(\log A)^2 + B(\log B)^2\right\} (A+B)^{\frac{1}{2}} - (A \log A + B \log B)^2 \\ &= \begin{pmatrix} 0.0132725\ldots & 0.012528\ldots & -0.00579579\ldots \\ 0.012528\ldots & 0.0161651\ldots & 0.00383114\ldots \\ -0.00579579\ldots & 0.00383114\ldots & 0.00777366\ldots \end{pmatrix} \end{aligned}$$

and the eigenvalues of X are $0.027386\ldots$, $0.0123316\ldots$ and $-0.00250627\ldots$, so that (1) does not satisfy and also by using (2.1), (2.2), (2.6) and (2.8) we have

$$\begin{aligned} Y &= \left\{A(\log A)^2 + B(\log B)^2\right\}^{\frac{1}{2}} (A+B) \left\{A(\log A)^2 + B(\log B)^2\right\}^{\frac{1}{2}} \\ &\quad -(A\log A + B\log B)^2 \\ &= \begin{pmatrix} 0.0125972\ldots & 0.0118891\ldots & -0.00806437\ldots \\ 0.0118891\ldots & 0.0244957\ldots & 0.000451054\ldots \\ -0.00806437\ldots & 0.000451054\ldots & 0.00011833\ldots \end{pmatrix} \end{aligned}$$

and the eigenvalues of Y are $0.0323307\ldots$, $0.0100147\ldots$, and $-0.00513409\ldots$, so that (2) does not satisfy. These two positive matrices $A \leqslant I$ and $B \leqslant I$ do not simultaneously satisfy (1) and (2).

3. Further problems on matrix inequalities related to the questions in Section 1

In this section we shall consider some related problems to the questions in Section 1 proposed by Yanagi–Furuichi–Kuriyama on trace inequalities.

Problem 1. We consider whether the following matrix inequalities

$$\begin{aligned} (p1) \qquad & (A+B)^{\frac{s}{2}} \left\{A(\log A)^2 + B(\log B)^2\right\} (A+B)^{\frac{s}{2}} \\ & \geqslant (A+B)^{\frac{s-1}{2}} (A\log A + B\log B)^2 (A+B)^{\frac{s-1}{2}}, \end{aligned}$$

$$\begin{aligned} (p2) \qquad & (A+B)^{\frac{s}{2}} \left\{A(\log A)^2 + B(\log B)^2\right\} (A+B)^{\frac{s}{2}} \\ & \geqslant (A\log A + B\log B)(A+B)^{s-1}(A\log A + B\log B), \end{aligned}$$

$$\begin{aligned} (p3) \qquad & \left\{A(\log A)^2 + B(\log B)^2\right\}^{\frac{1}{2}} (A+B)^s \left\{A(\log A)^2 + B(\log B)^2\right\}^{\frac{1}{2}} \\ & \geqslant (A+B)^{\frac{s-1}{2}} (A\log A + B\log B)^2 (A+B)^{\frac{s-1}{2}} \end{aligned}$$

and

$$\begin{aligned} (p4) \qquad & \left\{A(\log A)^2 + B(\log B)^2\right\}^{\frac{1}{2}} (A+B)^s \left\{A(\log A)^2 + B(\log B)^2\right\}^{\frac{1}{2}} \\ & \geqslant (A\log A + B\log B)(A+B)^{s-1}(A\log A + B\log B) \end{aligned}$$

for any two positive matrices $A \leqslant I$, $B \leqslant I$ and $s \in [0, 1]$ hold or not.

Remark 3.1. We remark that $(p2)$ in $s = 0$ just coincides with $(p4)$ in $s = 0$ and this is proved by [2] in the step of Theorem 2.4 in [2–(7)], that is,

$$\begin{aligned} (p4) \qquad & A(\log A)^2 + B(\log B)^2 \\ & \geqslant (A\log A + B\log B)(A+B)^{-1}(A\log A + B\log B) \end{aligned}$$

for any two positive matrices $A \leqslant I$, $B \leqslant I$.

Remark 3.2. We remark that $(p1)$ in $s = 1$ just coincides with $(p2)$ in $s = 1$, that is, this is (1) itself in Question in Section 1, also $(p3)$ in $s = 1$ just coincides with $(p4)$ in $s = 1$, that is, this is (2) itself in Question in Section 1, and a counterexample which do not simultaneously satisfy (1) and (2) is given in Section 2.

4. Several counterexamples to the further problems in Section 3

In this section we shall construct several counterexamples to the further problems in Section 3.

Counterexample 4.1. Two positive matrices $A \leqslant I$ and $B \leqslant I$ which do not satisfy (4.1).

$$\begin{aligned} &A(\log A)^2 + B(\log B)^2 \\ &\quad \geqslant (A+B)^{\frac{-1}{2}}(A\log A + B\log B)^2(A+B)^{\frac{-1}{2}}. \end{aligned} \tag{4.1}$$

We remark that (4.1) just corresponds to the case $(p1)$ in $s = 0$ and the case $(p3)$ in $s = 0$.

Next we take A and B as follows:

$$A = \begin{pmatrix} 0.0956175 & 0.0987594 & 0.122397 \\ 0.0987594 & 0.122116 & 0.154594 \\ 0.122397 & 0.154594 & 0.240864 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0.139939 & 0.148548 & 0.0960374 \\ 0.148548 & 0.174931 & 0.118901 \\ 0.0960374 & 0.118901 & 0.083098 \end{pmatrix}.$$

Then A and B are both positive definite matrices $A \leqslant I$ and $B \leqslant I$, in fact the eigenvalues of A are $0.422416\ldots$, $0.0292124\ldots$ and $0.00696822\ldots$, and also the eigenvalues of B are $0.384403\ldots$, $0.0133206\ldots$ and $0.000243971\ldots$,

$$\begin{aligned} X &= A(\log A)^2 + B(\log B)^2 - (A+B)^{\frac{-1}{2}}(A\log A + B\log B)^2(A+B)^{\frac{-1}{2}} \\ &= \begin{pmatrix} 0.00191909\ldots & 0.0341979\ldots & -0.0249438\ldots \\ 0.0341979\ldots & 0.00421238\ldots & -0.0496528\ldots \\ -0.0249438\ldots & -0.0496528\ldots & 0.104868\ldots \end{pmatrix} \end{aligned}$$

and the eigenvalues of X are $0.135234\ldots$, $-0.0339218\ldots$ and $0.00968665\ldots$, so that (4.1) does not satisfy.

Counterexample 4.2. Two positive matrices $A \leqslant I$ and $B \leqslant I$ which do not simultaneously satisfy (4.2) and (4.2′).

$$\begin{aligned}&(A+B)^{\frac{1}{4}}\left\{(A(\log A)^2+B(\log B)^2)\right\}(A+B)^{\frac{1}{4}}\\&\quad\geqslant (A+B)^{\frac{-1}{4}}(A\log A+B\log B)^2(A+B)^{\frac{-1}{4}},\end{aligned}\tag{4.2}$$

$$\begin{aligned}&(A+B)^{\frac{1}{4}}\left\{(A(\log A)^2+B(\log B)^2)\right\}(A+B)^{\frac{1}{4}}\\&\quad\geqslant (A\log A+B\log B)(A+B)^{\frac{-1}{2}}(A\log A+B\log B).\end{aligned}\tag{4.2$'$}$$

We remark that (4.2) just corresponds to the case $(p1)$ in $s=\frac{1}{2}$ and (4.2′) just corresponds to the case $(p2)$ in $s=\frac{1}{2}$.

Next we take A and B as follows:

$$A=\begin{pmatrix}0.126139 & 0.151703 & 0.206418\\ 0.151703 & 0.284223 & 0.345983\\ 0.206418 & 0.345983 & 0.509856\end{pmatrix}$$

and

$$B=\begin{pmatrix}0.262841 & 0.263463 & 0.215279\\ 0.263463 & 0.408157 & 0.338422\\ 0.215279 & 0.338422 & 0.281068\end{pmatrix}.$$

Then A and B are both positive definite matrices $A\leqslant I$ and $B\leqslant I$, in fact the eigenvalues of A are 0.851426…, 0.036208… and 0.0324711…, and also the eigenvalues of B are 0.877323…, 0.0745353… and 0.000207318…,

$$\begin{aligned}X&=(A+B)^{\frac{1}{4}}\left\{(A(\log A)^2+B(\log B)^2)\right\}(A+B)^{\frac{1}{4}}\\&\quad-(A+B)^{\frac{-1}{4}}(A\log A+B\log B)^2(A+B)^{\frac{-1}{4}}\\&=\begin{pmatrix}0.0267048\ldots & 0.0245234\ldots & -0.0276752\ldots\\ 0.0245234\ldots & 0.026167\ldots & -0.00884884\ldots\\ -0.0276752\ldots & -0.00884884\ldots & 0.0210792\ldots\end{pmatrix}\end{aligned}$$

and the eigenvalues of X are 0.0666133.., 0.0152701… and −0.00793239…, so that (4.2) does not satisfy.

$$\begin{aligned}Y&=(A+B)^{\frac{1}{4}}\{(A(\log A)^2+B(\log B)^2)\}(A+B)^{\frac{1}{4}}\\&\quad-(A\log A+B\log B)(A+B)^{\frac{-1}{2}}(A\log A+B\log B)\\&=\begin{pmatrix}0.028124\ldots & 0.0222034\ldots & -0.0231015\ldots\\ 0.0222034\ldots & 0.027459\ldots & -0.0105266\ldots\\ -0.0231015\ldots & -0.0105266\ldots & 0.0183679\ldots\end{pmatrix}\end{aligned}$$

and the eigenvalues of X are 0.0630956…, 0.0130865… and −0.00223121…, so that (4.2′) does not satisfy.

Counterexample 4.3. Two positive matrices $A\leqslant I$ and $B\leqslant I$ which do not simultaneously satisfy (4.3) and (4.3′).

$$\begin{aligned}&\left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}(A+B)^{\frac{1}{2}}\left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}\\&\quad\geqslant (A+B)^{\frac{-1}{4}}(A\log A+B\log B)^2(A+B)^{\frac{-1}{4}}\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}&\{A(\log A)^2+B(\log B)^2\}^{\frac{1}{2}}(A+B)^{\frac{1}{2}}\{A(\log A)^2+B(\log B)^2\}^{\frac{1}{2}}\\&\quad\geqslant (A\log A+B\log B)(A+B)^{\frac{-1}{2}}(A\log A+B\log B).\end{aligned}\tag{4.3$'$}$$

We remark that (4.3) just corresponds to the case $(p3)$ in $s=\frac{1}{2}$ and (4.3′) just corresponds to the case $(p4)$ in $s=\frac{1}{2}$.

Next we take A and B as follows:

$$A=\begin{pmatrix}0.0956175 & 0.0987594 & 0.122397\\ 0.0987594 & 0.122116 & 0.154594\\ 0.122397 & 0.154594 & 0.240864\end{pmatrix}$$

and

$$B=\begin{pmatrix}0.139939 & 0.148548 & 0.0960374\\ 0.148548 & 0.174931 & 0.118901\\ 0.0960374 & 0.118901 & 0.083098\end{pmatrix}.$$

Then A and B are both positive definite matrices $A\leqslant I$ and $B\leqslant I$, in fact the eigenvalues of A are $0.422416\ldots$, $0.0292124\ldots$ and $0.00696822\ldots$, and also the eigenvalues of B are $0.384403\ldots$, $0.0133206\ldots$ and $0.000243971\ldots$,

$$\begin{aligned}X&=\left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}(A+B)^{\frac{1}{2}}\left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}\\&\quad-(A+B)^{\frac{-1}{4}}(A\log A+B\log B)^2(A+B)^{\frac{-1}{4}}\\&=\begin{pmatrix}0.00533626\ldots & 0.00813644\ldots & -0.00905425\ldots\\ 0.00813644\ldots & 0.00864186\ldots & -0.00530569\ldots\\ -0.00905425\ldots & -0.00530569\ldots & 0.0213793\ldots\end{pmatrix}\end{aligned}$$

and the eigenvalues of X are $0.0289216\ldots$, $0.00849651\ldots$ and $-0.00206067\ldots$, so that (4.2) does not satisfy.

$$\begin{aligned}Y&=\left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}(A+B)^{\frac{1}{2}}\left\{A(\log A)^2+B(\log B)^2\right\}^{\frac{1}{2}}\\&\quad-(A\log A+B\log B)(A+B)^{\frac{-1}{2}}(A\log A+B\log B)\\&=\begin{pmatrix}0.00752669\ldots & 0.00910019\ldots & -0.0126332\ldots\\ 0.00910019\ldots & 0.0165803\ldots & -0.00388585\ldots\\ -0.0126332\ldots & -0.00388585\ldots & 0.0112505\ldots\end{pmatrix}\end{aligned}$$

and the eigenvalues of Y are $0.0288944\ldots$, $0.0108923\ldots$ and $-0.00442932\ldots$, so that (4.2′) does not satisfy.

We omit to describe the detail data in Counterexample 4.1, Counterexample 4.2 and Counterexample 4.3.

We used Mathematica 4.2 in order to calculate these data in this paper.

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